

Heisenberg Uncertainty Principle for the q -Bessel Fourier transform

Lazhar Dhaouadi *

Abstract

In this paper we use an I.I. Hirschman-W. Beckner entropy argument to give an uncertainty inequality for the q -Bessel Fourier transform:

$$\mathcal{F}_{q,v}f(x) = c_{q,v} \int_0^\infty f(t) j_v(xt, q^2) t^{2v+1} d_q t,$$

where $j_v(x, q)$ is the normalized Hahn-Exton q -Bessel function.

1 Introduction

I.I. Hirschman-W. Beckner entropy argument is one further variant of Heisenberg's uncertainty principle.

Let \widehat{f} be the Fourier transform of f defined by

$$\widehat{f}(x) = \int_{-\infty}^{\infty} f(y) e^{2i\pi xy} dy, \quad x \in \mathbb{R}.$$

If $f \in L^2(\mathbb{R})$ with L^2 -norm $\|f\|_2 = 1$, then by Plancherel's theorem $\|\widehat{f}\|_2 = 1$, so that $|f(x)|^2$ and $|\widehat{f}(x)|^2$ are probability frequency functions. The variance of a probability frequency g is defined by

$$V[g] = \int_{\mathbb{R}} x^2 g(x) dx.$$

The Heisenberg uncertainty principle can be stated as follows

$$V[|f|^2] V[|\widehat{f}|^2] \geq \frac{1}{16\pi^2}. \quad (1)$$

*Institut Préparatoire aux Etudes d'Ingénieur de Bizerte (Université du 7 novembre Carthage). Route Menzel Abderrahmene Bizerte, 7021 Zarzouna, Tunisia. E-mail lazhardhaouadi@yahoo.fr

If g is a probability frequency function, then the entropy of g is defined by

$$E(g) = \int_{\mathbb{R}} g(x) \log(x) dx.$$

With f as above, Hirschman [10] proved that

$$E(|f|^2) + E(|\hat{f}|^2) \leq 0. \quad (2)$$

By an inequality of Shannon and Weaver it follows that (2) implies (1). Using the Babenko-Beckner inequality

$$\|\hat{f}\|_{p'} \leq A(p) \|f\|_p, \quad 1 < p < 2, \quad A(p) = \left[p^{1/p} (p')^{-1/p'} \right]^{1/2},$$

in Hirschman's proof of (2) another uncertainty inequality is deduced. For more detail the reader can consult [8,10,11].

In this paper we use I.I. Hirschman entropy argument to give an uncertainty inequality for the q -Bessel Fourier transform (also called q -Hankel transform).

Note that other versions of the Heisenberg uncertainty principle for the q -Fourier transform have recently appeared in the literature [1,2,6]. There are some differences of the results cited above and our result:

- In [1] the uncertainty inequality is established for the q -cosine and q -sine transform but here is established for the q -Bessel transform.
- In [2] the uncertainty inequality is for the q^2 -Fourier transform but here is for the q -Hankel transform.
- In [6] the uncertainty inequality is established for functions in q -Schwartz space. In this paper the uncertainty inequality is established for functions in $\mathcal{L}_{q,2,v}$ space.

The inequality discussed here is a quantitative uncertainty principle which gives information about how a function and its q -Bessel Fourier transform relate. Qualitative uncertainty principles give information about how a function (and its Fourier transform) behave under certain circumstances. A classical qualitative uncertainty principle called Hardy's theorem. In [4,7] a q -version of the Hardy's theorem for the q -Bessel Fourier transform was established.

In the end, our objective is to develop a coherent harmonic analysis attached to the q -Bessel operator

$$\Delta_{q,v} f(x) = \frac{1}{x^2} [f(q^{-1}x) - (1 + q^{2v})f(x) + q^{2v}f(qx)].$$

Thus, this paper is an opportunity to implement the arguments of the q -Bessel Fourier analysis proved before, as the Plancherel formula, the positivity of the q -translation operator, the q -convolution product, the q -Gauss kernel...

2 The q -Bessel Fourier transform

In the following we will always assume $0 < q < 1$ and $v > -1$. We denote by

$$\mathbb{R}_q = \{\pm q^n, \quad n \in \mathbb{Z}\}, \quad \mathbb{R}_q^+ = \{q^n, \quad n \in \mathbb{Z}\}.$$

For more informations on the q -series theory the reader can see the references [9,12,14] and the references [3,5,13] about the q -bessel Fourier analysis. Also for details of the proofs of the following results in this section can be found in [3].

Definition 1 *The q -Bessel operator is defined as follows*

$$\Delta_{q,v}f(x) = \frac{1}{x^2} [f(q^{-1}x) - (1 + q^{2v})f(x) + q^{2v}f(qx)].$$

Definition 2 *The normalized q -Bessel function of Hahn-Exton is defined by*

$$j_v(x, q^2) = \sum_{n=0}^{\infty} (-1)^n \frac{q^{n(n+1)}}{(q^{2v+2}, q^2)_n (q^2, q^2)_n} x^{2n}.$$

Proposition 1 *The function*

$$x \mapsto j_v(\lambda x, q^2)$$

is the eigenfunction of the operator $\Delta_{q,v}$ associated with the eigenvalue $-\lambda^2$.

Definition 3 *The q -Jackson integral of a function f defined on \mathbb{R}_q is*

$$\int_0^\infty f(t) d_q t = (1 - q) \sum_{n \in \mathbb{Z}} q^n f(q^n).$$

Definition 4 *We denote by $\mathcal{L}_{q,p,v}$ the space of even functions f defined on \mathbb{R}_q such that*

$$\|f\|_{q,p,v} = \left[\int_0^\infty |f(x)|^p x^{2v+1} d_q x \right]^{1/p} < \infty.$$

Definition 5 We denote by $\mathcal{C}_{q,0}$ the space of even functions defined on \mathbb{R}_q tending to 0 as $x \rightarrow \pm\infty$ and continuous at 0 equipped with the topology of uniform convergence. The space $\mathcal{C}_{q,0}$ is complete with respect to the norm

$$\|f\|_{q,\infty} = \sup_{x \in \mathbb{R}_q} |f(x)|.$$

Definition 6 The q -Bessel Fourier transform $\mathcal{F}_{q,v}$ (also called q -Hankel transform) is defined by

$$\mathcal{F}_{q,v}f(x) = c_{q,v} \int_0^\infty f(t) j_v(xt, q^2) t^{2v+1} d_q t, \quad \forall x \in \mathbb{R}_q.$$

where

$$c_{q,v} = \frac{1}{1-q} \frac{(q^{2v+2}; q^2)_\infty}{(q^2; q^2)_\infty}.$$

Proposition 2 Let $f \in \mathcal{L}_{q,1,v}$ then $\mathcal{F}_{q,v}f$ existe and $\mathcal{F}_{q,v}f \in \mathcal{C}_{q,0}$.

Definition 7 The q -translation operator is given as follows

$$T_{q,x}^v f(y) = c_{q,v} \int_0^\infty \mathcal{F}_{q,v}f(t) j_v(yt, q^2) j_v(xt, q^2) t^{2v+1} d_q t \quad \forall f \in \mathcal{L}_{q,1,v}.$$

Definition 8 The operator $T_{q,x}^v$ is said positive if $T_{q,x}^v f \geq 0$ when $f \geq 0$ for all $x \in \mathbb{R}_q$. We denote by Q_v the domain of positivity of $T_{q,x}^v$

$$Q_v = \{q \in]0, 1[, \quad T_{q,x}^v \text{ is positive}\}.$$

In the following we assume that $q \in Q_v$.

Proposition 3 If $f \in \mathcal{L}_{q,1,v}$ then

$$\int_0^\infty T_{q,x}^v f(y) y^{2v+1} d_q y = \int_0^\infty f(y) y^{2v+1} d_q y.$$

Definition 9 The q -convolution product is defined as follows

$$f *_q g(x) = c_{q,v} \int_0^\infty T_{q,x}^v f(y) g(y) y^{2v+1} d_q y.$$

Proposition 4 Let $f, g \in \mathcal{L}_{q,1,v}$ then $f *_q g \in \mathcal{L}_{q,1,v}$ and we have

$$\mathcal{F}_{q,v}(f *_q g) = \mathcal{F}_{q,v}(g) \times \mathcal{F}_{q,v}(f).$$

Proposition 5 Let $f \in \mathcal{L}_{q,1,v}$ and $g \in \mathcal{L}_{q,2,v}$ then $f *_q g \in \mathcal{L}_{q,2,v}$ and we have

$$\mathcal{F}_{q,v}(f *_q g) = \mathcal{F}_{q,v}(f) \times \mathcal{F}_{q,v}(g).$$

Theorem 1 The q -Bessel Fourier transform $\mathcal{F}_{q,v}$ satisfies

1. $\mathcal{F}_{q,v}$ sends $\mathcal{L}_{q,2,v}$ to $\mathcal{L}_{q,2,v}$.
2. For $f \in \mathcal{L}_{q,2,v}$, we have

$$\|\mathcal{F}_{q,v}(f)\|_{q,2,v} = \|f\|_{q,2,v}.$$

3. The operator $\mathcal{F}_{q,v} : \mathcal{L}_{q,2,v} \rightarrow \mathcal{L}_{q,2,v}$ is bijective and

$$\mathcal{F}_{q,v}^{-1} = \mathcal{F}_{q,v}.$$

Proposition 6 Given $1 < p \leq 2$ and $\frac{1}{p} + \frac{1}{\bar{p}} = 1$. If $f \in \mathcal{L}_{q,p,v}$ then

$$\mathcal{F}_{q,v}(f) \in \mathcal{L}_{\bar{p},2,v}$$

and

$$\|\mathcal{F}_{q,v}(f)\|_{q,\bar{p},v} \leq B_{q,v}^{(\frac{2}{\bar{p}}-1)} \|f\|_{q,p,v},$$

where

$$B_{q,v} = \frac{1}{1-q} \frac{(-q^2; q^2)_\infty (-q^{2v+2}; q^2)_\infty}{(q^2; q^2)_\infty}.$$

Definition 10 The q -exponential function is defined by

$$e(z, q) = \sum_{n=0}^{\infty} \frac{z^n}{(q, q)_n} = \frac{1}{(z; q)_\infty}, \quad |z| < 1.$$

Proposition 7 The q -Gauss kernel

$$G^v(x, t^2, q^2) = \frac{(-q^{2v+2}t^2, -q^{-2v}/t^2; q^2)_\infty}{(-t^2, -q^2/t^2; q^2)_\infty} e\left(-\frac{q^{-2v}}{t^2}x^2, q^2\right), \quad \forall x, t \in \mathbb{R}_q^+$$

satisfies

$$\mathcal{F}_{q,v} \{e(-t^2y^2, q^2)\} (x) = G^v(x, t^2, q^2),$$

and for all function $f \in \mathcal{L}_{q,2,v}$

$$\lim_{n \rightarrow \infty} \|G^v(x, q^{2n}, q^2) *_q f - f\|_{q,2,v} = 0.$$

3 Uncertainty Principle

The following Lemma are crucial for the proof of our main result. First we enunciate the Jensens inequality

Lemma 1 *Let γ be a probability measure on \mathbb{R}_q^+ . Let g be a convex function on a subset I of \mathbb{R} . If $\psi : \mathbb{R}_q^+ \rightarrow I$ satisfies*

$$\int_0^\infty \psi(u) d\gamma(u) \in I,$$

then we have

$$g\left(\int_0^\infty \psi(x) d\gamma(x)\right) \leq \int_0^\infty g \circ \psi(x) d\gamma(x).$$

Proof. Let

$$t = \int_0^\infty \psi(u) d\gamma(u).$$

There exist $c \in \mathbb{R}$ such that for all $y \in I$ it holds

$$g(y) \geq g(t) + c(y - t).$$

Now let $y = \psi(x)$ we obtain

$$g(\psi(x)) \geq g(t) + c(\psi(x) - t).$$

Integrating both sides and using the special value of t gives

$$\int_0^\infty g(\psi(x)) d\gamma(x) \geq \int_0^\infty [g(t) + c(\psi(x) - t)] d\gamma(x) = g(t).$$

This finish the proof. ■

Lemma 2 *Let f be an even function defined on \mathbb{R}_q . Assume $\psi : \mathbb{R} \rightarrow \mathbb{R}_+$ is a convex function and $\psi \circ f \in \mathcal{L}_{q,1,v}$. If ϱ_n is a sequence of non-negative function in $\mathcal{L}_{q,1,v}$ such that*

$$\mathcal{F}_{q,v}(\varrho_n)(0) = c_{q,v} \int_0^\infty \varrho_n(x) x^{2v+1} d_q x = 1$$

*and $\varrho_n *_q f \rightarrow f$ then $\psi \circ (\varrho_n *_q f)$ is in $\mathcal{L}_{q,1,v}$ and*

$$\lim_{n \rightarrow \infty} \int_0^\infty \psi \circ (\varrho_n *_q f)(x) x^{2v+1} d_q x = \int_0^\infty \psi \circ f(x) x^{2v+1} d_q x.$$

Proof. For a given x and by Proposition 3 we have

$$c_{q,v} \int_0^\infty T_{q,x}^v \varrho_n(y) y^{2v+1} d_q y = 1$$

From the positivity of $T_{q,x}^v$ we see that

$$c_{q,v} T_{q,x}^v \varrho_n(y) y^{2v+1} d_q y$$

is a probability measure on \mathbb{R}_q^+ . The following holds by Jensens Inequality

$$\begin{aligned} \psi \circ (\varrho_n *_q f)(x) &= \psi \left[c_{q,v} \int_0^\infty f(y) T_{q,x}^v \varrho_n(y) y^{2v+1} d_q y \right] \\ &\leq c_{q,v} \int_0^\infty \psi \circ f(y) T_{q,x}^v \varrho_n(y) y^{2v+1} d_q y \\ &= \varrho_n *_q \psi \circ f(x). \end{aligned}$$

By the use of the Fatou's Lemma and Proposition 4 we obtain

$$\begin{aligned} &\int_0^\infty \psi \circ f(x) x^{2v+1} d_q x \\ &= \int_0^\infty \liminf_{n \rightarrow \infty} \psi \circ (\varrho_n *_q f)(x) x^{2v+1} d_q x \\ &\leq \liminf_{n \rightarrow \infty} \int_0^\infty \psi \circ (\varrho_n *_q f)(x) x^{2v+1} d_q x \\ &\leq \limsup_{n \rightarrow \infty} \int_0^\infty \psi \circ (\varrho_n *_q f)(x) x^{2v+1} d_q x \\ &\leq \lim_{n \rightarrow \infty} \int_0^\infty \varrho_n *_q \psi \circ f(x) x^{2v+1} d_q x \\ &= \frac{1}{c_{q,v}} \lim_{n \rightarrow \infty} \mathcal{F}_{q,v}(\varrho_n)(0) \times \mathcal{F}_{q,v}(\psi \circ f)(0) \\ &= \int_0^\infty \psi \circ f(x) x^{2v+1} d_q x. \end{aligned}$$

This finish the proof. ■

Definition 11 For a positive function ϕ define the entropy of ϕ to be

$$E(\phi) = \int_0^\infty \phi(x) \log \phi(x) x^{2v+1} d_q x.$$

$E(\phi)$ can have values in $[-\infty, \infty]$.

Remark 1 For a given $c \in \mathbb{R}_q^+$ let

$$d\gamma(x) = k_c^{-1} \exp(-|cx|^a) x^{2v+1} d_q x$$

where

$$\sigma_a = \int_0^\infty \exp(-|x|^a) x^{2v+1} d_q x, \quad k_c = \frac{\sigma_a}{c^{2v+2}}.$$

Then $d\gamma(x)$ is a probability measure on \mathbb{R}_q^+ .

Lemma 3 Let $a > 0$. For a positive function $\phi \in \mathcal{L}_{q,1,v}$ such that

$$\|\phi\|_{q,1,v} = 1$$

and

$$M_a(\phi) = \left(\int_0^\infty |x|^a \phi(x) x^{2v+1} d_q x \right)^{\frac{1}{a}}$$

is finite, we have

$$-E(\phi) \leq \log k_c + c^a M_a^a(\phi). \quad (3)$$

Proof. Indeed, defining

$$\psi(x) = k_c \exp(|cx|^a) \phi(x),$$

From Remark 1 we see that

$$\int_0^\infty \psi(x) d\gamma(x) = 1.$$

According to the fact that $g : t \mapsto t \log t$ is convex on \mathbb{R}_+^* , so Jensen's inequality gives

$$g \left[\int_0^\infty \psi(x) d\gamma(x) \right] \leq \int_0^\infty g \circ \psi(x) d\gamma(x).$$

Hence,

$$0 = \left[\int_0^\infty \psi(x) d\gamma(x) \right] \log \left[\int_0^\infty \psi(x) d\gamma(x) \right] \leq \int_0^\infty \psi(x) \log \psi(x) d\gamma(x).$$

This implies

$$\begin{aligned} 0 &\leq \int_0^\infty \phi(x) \log [k_c \exp(|cx|^a) \phi(x)] x^{2v+1} d_q x \\ &= \int_0^\infty \phi(x) [\log k_c + |cx|^a + \log \phi(x)] x^{2v+1} d_q x. \end{aligned}$$

$$0 \leq \log k_c + c^a \int_0^\infty |x|^a \phi(x) x^{2v+1} d_q x + \int_0^\infty \phi(x) \log \phi(x) x^{2v+1} d_q x.$$

In the end

$$0 \leq \log k_c + c^a M_a^a(\phi) + E(\phi).$$

This finish the proof. ■

Lemma 4 *Let $f \in \mathcal{L}_{q,1,v} \cap \mathcal{L}_{q,2,v}$ then we have*

$$E(|f|^2) + E(|\mathcal{F}_{q,v} f|^2) \leq 2 \|f\|_{q,v,2}^2 \log(B_{q,v} \|f\|_{q,v,2}^2). \quad (4)$$

Proof. Hölder inequality implies that f will be in $\mathcal{L}_{q,p,v}$ for $1 < p \leq 2$. With

$$\frac{1}{p} + \frac{1}{\bar{p}} = 1,$$

Hausdorff-Young's inequality (Proposition 6) tells us that $\mathcal{F}_{q,v} f$ is in $\mathcal{L}_{q,\bar{p},v}$. So we can define the functions

$$A(p) = \int_0^\infty |f(x)|^p d_q x \quad \text{and} \quad B(\bar{p}) = \int_0^\infty |\mathcal{F}_{q,v} f(x)|^{\bar{p}} x^{2v+1} d_q x.$$

Now define

$$\begin{aligned} C(p) &= \log \|\mathcal{F}_{q,v} f\|_{q,\bar{p},v} - \log \left(B_{q,v}^{\frac{2}{p}-1} \|f\|_{q,p,v} \right) \\ &= \frac{1}{\bar{p}} \log B(\bar{p}) - \frac{1}{p} \log A(p) - \left(\frac{2}{p} - 1 \right) \log B_{q,v}. \end{aligned}$$

By Hausdorff-Young's inequality

$$C(p) \leq 0, \quad \text{for } 1 < p < 2,$$

and by Plancherel equality (Theorem 1 part 2)

$$C(2) = 0.$$

Then

$$C'(2^-) \geq 0.$$

On the other hand for $1 < p < 2$ we have

$$C'(p) = \frac{\bar{p}'}{\bar{p}} \frac{B'(\bar{p})}{B(\bar{p})} - \frac{\bar{p}'}{\bar{p}^2} \log B(\bar{p}) - \frac{1}{p} \frac{A'(p)}{A(p)} + \frac{1}{p^2} \log A(p) + \frac{2}{p^2} \log B_{q,v}.$$

The derivative of \bar{p} with respect to p is

$$\bar{p}' = -\frac{1}{(p-1)^2}.$$

For a given $x > 0$ we have

$$\lim_{p \rightarrow 2} \frac{x^p - x^2}{p - 2} = x^2 \log x.$$

Then

$$A'(2^-) = \lim_{p \rightarrow 2^-} \frac{A(p) - A(2)}{p - 2} = \frac{1}{2} E(|f|^2),$$

$$B'(2^+) = \lim_{\bar{p} \rightarrow 2^+} \frac{B(\bar{p}) - B(2)}{\bar{p} - 2} = \frac{1}{2} E(|\mathcal{F}_{q,v} f|^2).$$

Since

$$p \mapsto \frac{x^p - x^2}{p - 2}$$

is an increasing function, the exchange of the signs limit and integral is valid sense. On the other hand

$$\lim_{p \rightarrow 2^-} A(p) = \|f\|_{q,v,2}^2, \quad \lim_{\bar{p} \rightarrow 2^+} B(p) = \|\mathcal{F}_{q,v} f\|_{q,v,2}^2 = \|f\|_{q,v,2}^2.$$

So

$$C'(2^-) = \lim_{p \rightarrow 2^-} \frac{C(p) - C(2)}{p - 2} = -\frac{1}{2\|f\|_{q,v,2}^2} [A'(2^-) + B'(2^+)] + \frac{1}{2} \log(B_{q,v}\|f\|_{q,v,2}^2).$$

Therefore

$$A'(2^-) + B'(2^+) - \|f\|_{q,v,2}^2 \log(B_{q,v}\|f\|_{q,v,2}^2) \leq 0,$$

and then

$$E(|f|^2) + E(|\mathcal{F}_{q,v} f|^2) \leq 2\|f\|_{q,v,2}^2 \log(B_{q,v}\|f\|_{q,v,2}^2).$$

This finish the proof. ■

Lemma 5 *Let $f \in \mathcal{L}_{q,2,v}$ then we have*

$$E(|f|^2) + E(|\mathcal{F}_{q,v} f|^2) \leq 2\|f\|_{q,v,2}^2 \log(B_{q,v}\|f\|_{q,v,2}^2). \quad (5)$$

Proof. Assume that $E(|f|^2)$ and $E(|\mathcal{F}_{q,v}f|^2)$ are defined and then approximate f by functions in $\mathcal{L}_{q,1,v} \cap \mathcal{L}_{q,2,v}$. Let

$$h_n(x) = e(-q^{2n}x^2, q^2).$$

The function h_n is in $\mathcal{L}_{q,2,v}$ then $h_nf \in \mathcal{L}_{q,1,v}$. On the other hand $h_n \in \mathcal{C}_{q,0}$ then $h_nf \in \mathcal{L}_{q,2,v}$. We obtain

$$h_nf \in \mathcal{L}_{q,1,v} \cap \mathcal{L}_{q,2,v}.$$

The following holds by (2)

$$E(|h_nf|^2) + E(|\mathcal{F}_{q,v}(h_nf)|^2) \leq 2\|h_nf\|_{q,2,v}^2 \log(B_{q,v}\|h_nf\|_{q,2,v}^2). \quad (6)$$

One can see by the Lebesgue Dominated Convergence Theorem that

$$\lim_{n \rightarrow \infty} \|h_nf\|_{q,2,v} = \|f\|_{q,2,v} \quad (7)$$

and

$$\lim_{n \rightarrow \infty} E(|h_nf|^2) = E(|f|^2). \quad (8)$$

By the use of Proposition 5 and the inversion formula (Theorem 1 part 3) we see that

$$\mathcal{F}_{q,v}(h_nf) = \mathcal{F}_{q,v}h_n *_q \mathcal{F}_{q,v}f.$$

We will prove that

$$\lim_{n \rightarrow \infty} E(|\mathcal{F}_{q,v}h_n *_q \mathcal{F}_{q,v}f|^2) = E(|\mathcal{F}_{q,v}f|^2).$$

The functions

$$\phi_1(x) = x^2 \log^+ |x| \text{ and } \phi_2(x) = x^2 \left(-\log^- |x| + \frac{3}{2} \right),$$

are convex on \mathbb{R} , where

$$\log^+ x = \max\{0, \log x\} \text{ and } \log^- x = \min\{0, \log x\}.$$

Note that

$$2\phi_1(x) - 2\phi_2(x) + 3x^2 = x^2 \log |x|^2.$$

Since

- From the inversion formula we see that

$$c_{q,v} \int_0^\infty \mathcal{F}_{q,v}h_n(t) t^{2v+1} d_q t = h_n(0) = 1.$$

- The function $\mathcal{F}_{q,v}h_n \geq 0$.
- The functions ϕ_i are convex on \mathbb{R} .
- $E(\mathcal{F}_{q,v}f)$ is finite then $\phi_i(\mathcal{F}_{q,v}f)$ is in $\mathcal{L}_{q,1,v}$.
- From Proposition 7 we have

$$\lim_{n \rightarrow \infty} \mathcal{F}_{q,v}h_n *_q \mathcal{F}_{q,v}f(x) = \mathcal{F}_{q,v}f(x)$$

we deduce that $\mathcal{F}_{q,v}h_n$ and ϕ_i satisfy the conditions of Lemma 2. Then we obtain

$$\lim_{n \rightarrow \infty} \int_0^\infty \phi_i \circ (\mathcal{F}_{q,v}h_n *_q \mathcal{F}_{q,v}f)(x) x^{2v+1} d_q x = \int_0^\infty \phi_i \circ (\mathcal{F}_{q,v}f)(x) x^{2v+1} d_q x, \quad i = 1, 2.$$

It also hold

$$\begin{aligned} E(|\mathcal{F}_{q,v}f|^2) &= 2 \int_0^\infty \phi_1(\mathcal{F}_{q,v}f) x^{2v+1} d_q \\ &\quad - 2 \int_0^\infty \phi_2(\mathcal{F}_{q,v}f) x^{2v+1} d_q x + 3 \|\mathcal{F}_{q,v}f\|_{q,2,v}^2, \end{aligned}$$

and

$$\begin{aligned} E(|\mathcal{F}_{q,v}h_n *_q \mathcal{F}_{q,v}f|^2) &= 2 \int_0^\infty \phi_1(\mathcal{F}_{q,v}h_n *_q \mathcal{F}_{q,v}f) x^{2v+1} d_q x \\ &\quad - 2 \int_0^\infty \phi_2(\mathcal{F}_{q,v}h_n *_q \mathcal{F}_{q,v}f) x^{2v+1} d_q x \\ &\quad + 3 \|\mathcal{F}_{q,v}h_n *_q \mathcal{F}_{q,v}f\|_{q,2,v}^2. \end{aligned}$$

Then

$$\lim_{n \rightarrow \infty} E(|\mathcal{F}_{q,v}h_n *_q \mathcal{F}_{q,v}f|^2) = E(|\mathcal{F}_{q,v}f|^2). \quad (9)$$

With (6) and the limits (7), (8) and (9) we complete the proof of (5).

Note that these limits also hold in the case where $E(|f|^2)$ and $E(|\mathcal{F}_{q,v}f|^2)$ are ∞ or $-\infty$. ■

Now we are in position to state and prove the uncertainty inequality for the q -Bessel Fourier transform.

Theorem 2 *Given $a, b > 0$. Then for all $c, d \in \mathbb{R}_q^+$ satisfying*

$$0 < B_{q,v}^2 \frac{\sigma_a \sigma_b}{(cd)^{2v+2}} < 1,$$

the following hold for any function $f \in \mathcal{L}_{q,2,v}$

$$c^a \left\| x^{a/2} f \right\|_{q,2,v}^2 + d^b \left\| x^{b/2} \mathcal{F}_{q,v}f \right\|_{q,2,v}^2 \geq -\log \left(B_{q,v}^2 \frac{\sigma_a \sigma_b}{(cd)^{2v+2}} \right) \|f\|_{q,2,v}^2.$$

Proof. Assume that $\|f\|_{q,2,v} = 1$. By (3) we can write

$$\begin{aligned} -E(|f|^2) &\leq \log k_c + c^a \left\| x^{a/2} f \right\|_{q,2,v}^2 \\ -E\left(|\mathcal{F}_{q,v} f|^2\right) &\leq \log k_d + d^b \left\| x^{b/2} \mathcal{F}_{q,v} f \right\|_{q,2,v}^2. \end{aligned}$$

Which implies with (5)

$$\begin{aligned} -2 \log B_{q,v} &\leq -E(|f|^2) - E(|\mathcal{F}_{q,v} f|^2) \\ &\leq \log(k_c k_d) + c^a \left\| x^{a/2} f \right\|_{q,2,v}^2 + d^b \left\| x^{b/2} \mathcal{F}_{q,v} f \right\|_{q,2,v}^2. \end{aligned}$$

By replacing f by $\frac{f}{\|f\|_{q,2,v}}$ we get

$$c^a \left\| x^{a/2} f \right\|_{q,2,v}^2 + d^b \left\| x^{b/2} \mathcal{F}_{q,v} f \right\|_{q,2,v}^2 \geq -\log(B_{q,v}^2 k_c k_d) \|f\|_{q,2,v}^2.$$

This finish the proof. ■

Corollary 1 *There exist $k > 0$ such that for any function $f \in \mathcal{L}_{q,2,v}$ we have*

$$\|xf\|_{q,2,v} \|x\mathcal{F}_{q,v} f\|_{q,2,v} \geq k \|f\|_{q,2,v}^2.$$

Proof. Let $a = b = 2$ and $c = d$ then by Theorem 3

$$\|xf\|_{q,2,v}^2 + \|x\mathcal{F}_{q,v} f\|_{q,2,v}^2 \geq -\frac{1}{c^2} \log\left(B_{q,v}^2 \frac{\sigma_2^2}{c^{4(v+1)}}\right) \|f\|_{q,2,v}^2,$$

where

$$0 < \left(B_{q,v}^2 \frac{\sigma_2^2}{c^{4(v+1)}}\right) < 1.$$

Now put

$$f_t(x) = f(tx), \quad t \in \mathbb{R}_q^+,$$

then

$$\mathcal{F}_{q,v} f_t(x) = \frac{1}{t^{2v+2}} \mathcal{F}_{q,v} f(x/t), \quad \|x\mathcal{F}_{q,v} f_t\|_{q,2,v}^2 = \frac{1}{t^{2v}} \|\mathcal{F}_{q,v} f\|_{q,2,v}^2,$$

and

$$\|f_t\|_{q,2,v}^2 = \frac{1}{t^{2v+2}} \|f\|_{q,2,v}^2, \quad \|xf_t\|_{q,2,v}^2 = \frac{1}{t^{2v+4}} \|xf\|_{q,v,2}^2,$$

which gives

$$t^4 \|x\mathcal{F}_{q,v}f\|_{q,2,v}^2 + t^2 \frac{1}{c^2} \log \left(B_{q,v}^2 \frac{\sigma_2^2}{c^{4(v+1)}} \right) \|f\|_{q,v,2}^2 + \|xf\|_{q,2,v}^2 \geq 0,$$

and then

$$\|xf\|_{q,2,v} \|x\mathcal{F}_{q,v}f\|_{q,2,v} \geq \psi(c) \|f\|_{q,2,v}^2.$$

where

$$\psi(c) = \frac{v+1}{[\sigma_2 B_{q,v}]^{\frac{1}{v+1}}} |z_c \log(z_c)|, \quad z_c = \frac{[\sigma_2 B_{q,v}]^{\frac{1}{v+1}}}{c^2}, \quad 0 < z_c < 1.$$

One can see that

$$\sup_{0 < z_c < 1} \psi(c) = \psi(q^\alpha), \quad \alpha = \frac{\log[\sigma_2 B_{q,v}]}{2(1+v) \log q} + \frac{1}{2 \log q}.$$

Let

$$n_1 = \lfloor \alpha \rfloor, \quad n_2 = \lceil \alpha \rceil,$$

where $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$ are respectively the floor and ceiling functions. Now the constant k is given as follows

$$k = \psi(q^{n_1}), \quad \text{if } \lceil \alpha \rceil \geq \alpha - \frac{1}{2 \log q}$$

and

$$k = \max\{\psi(q^{n_1}), \psi(q^{n_2})\}, \quad \text{if } \lceil \alpha \rceil < \alpha - \frac{1}{2 \log q}.$$

This finish the proof. ■

References

- [1] N. Bettaibi, A. Fitouhi and W. Binous, Uncertainty principles for the q-trigonometric Fourier transforms, Math. Sci. Res. J. 11 (2007).
- [2] N. Bettaibi, Uncertainty principles in q^2 -analogue Fourier analysis, Math. Sci. Res. J. 11 (2007).
- [3] L. Dhaouadi, A. Fitouhi and J. El Kamel, Inequalities in q-Fourier Analysis, Journal of Inequalities in Pure and Applied Mathematics, Volume 7, Issue 5, Article 171, 2006.

- [4] L. Dhaouadi, Hardy's theorem for the q -Bessel Fourier transform, arXiv: 0707.2346 v1 [math.CA].
- [5] A. Fitouhi, M. Hamza and F. Bouzeffour, The $q - j_\alpha$ Bessel function J. Appr. Theory. 115, 144-166 (2002).
- [6] A. Fitouhi, N. Bettaibi, W. Binous and H.B. Elmonser, Uncertainty principles for the basic Bessel transform, Ramanujan J., in press.
- [7] A. Fitouhi, N. Bettaibi and R. Bettaieb, On Hardy's inequality for symmetric integral transforms and analogous, Appl. Math. Comput. 198 (2008).
- [8] G.B. Folland and A. Sitaram. The uncertainty principle: a mathematical survey. J. Fourier Anal. Appl., 3(3):207–238, 1997.
- [9] G. Gasper and M. Rahman, Basic hypergeometric series, Encyclopedia of mathematics and its applications 35, Cambridge university press, 1990.
- [10] I.I. Hirschman, Jr. A note on entropy. Amer. J. Math., 79:152156, 1957.
- [11] H.P. Heinig, and M. Smith, Extensions of the Heisenberg-Weyl inequality. Internat. J. Math. Math. Sci. 9 (1986), no. 1, 185–192.
- [12] F. H. Jackson, On a q -Definite Integrals, Quarterly Journal of Pure and Application Mathematics 41, 1910, 193-203.
- [13] T. H. Koornwinder and R. F. Swarttouw, On q -Analogues of the Hankel and Fourier Transform, Trans. A. M. S. 1992, 333, 445-461.
- [14] R. F. Swarttouw, The Hahn-Exton q -Bessel functions PhD Thesis The Technical University of Delft (1992).